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ON REGULAR REFLECTION OF A SHOCK WAVE FROM A RIGID WALL*

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The three-dimensional problem of reflection of a shock wave of arbitrary front shape from a fixed rigid wall is considered. Existence of a piecewise analytic solution of the problem defining the initial stage of regular reflection is proved. Expansion of solution behind the reflected wave front in converging power series in the neighborhood of the incident wave (of the moving along the wall intersection line of the incident wave front and the rigid wall) is obtained for prolonged instants of time. It is shown that such expansions generally occur only then, when the length of the trace velocity vector relative to gas behind the reflected wave projected on the plane normal to the trace exceeds the speed of sound. A bibliography of publications dealing with the problem of shock wave reflection can be found in /1,2/.

1. Statement of the problem. The motion of an inviscid non-heat-conducting gas is considered in the region of $\varphi(\mathbf{x}) \geqslant 0$ with surface $\Gamma_1 : \varphi(\mathbf{x}) = 0$ $(\nabla\varphi \neq 0)$ assumed to be a rigid impervious wall at which the gas velocity vector $\mathbf{u}$ satisfies the relation

$$\mathbf{u}\nabla\varphi = 0, \quad \mathbf{x} = (x, y, z) \in \Gamma_1 \tag{1.1}$$

Let the piecewise-analytic solution of gasdynamics equations

$$\rho\frac{d\mathbf{u}}{dt} + \nabla p = 0, \quad \frac{dp}{dt} + \rho c^2 \operatorname{div}\mathbf{u} = 0, \quad \frac{ds}{dt} = 0, \quad \rho = \psi(p, s) \tag{1.2}$$

defining shock wave propagation toward a rigid wall when $t \leqslant 0$, be known. Here p is the pressure, ρ is the density, s is the entropy, c is the speed of sound, and $\psi(p, s)$ is an analytic function that specifies the equation of state of a normal gas /3,4/. This means that ahead and behind the shock wave front $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$, $s(\mathbf{x}, t)$ are analytic functions of their arguments, and the shock wave surface Γ_2 is an analytic hypersurface in the four-dimensional space $\mathbf{x}$, t. The Hugoniot relations

$$[\rho v_n] = 0, \quad [p + \rho v_n^2] = 0, \quad [\varepsilon + p\rho^{-1} + {}^1/_2 v_n^2] = 0, \quad [\mathbf{u}_\sigma] = 0 \tag{1.3}$$

and the condition of entropy increase are satisfied on Γ_2. Here $[f]$ denotes the jump of quantity f at transition through the discontinuity, $v_n = D_n - u_n$, u_n and D_n are the velocities of gas and of shock wave front, respectively, in the direction of the normal $\mathbf{n}$ to the front, $\mathbf{u}_\sigma$ is the tangent velocity component, and ε is the specific internal energy. The solution ahead of the front satisfies condition (1.4). At instant $t = 0$ the shock wave front reaches the wall touching it at point Q. Further motion of gas has to be defined.

When $t > 0$ the solution structure changes, a shock wave reflected from the wall makes its appearance. The region of determination of the generalized gas motion in the space $\mathbf{x}$, t consists of three subregions, viz., region Ω_1 bounded by Γ_1, the incident wave surface Γ_2 and the plane $t = 0$; region Ω_2 bounded by Γ_2, the reflected wave surface Γ_3 and the plane $t = 0$; and region Ω_3 bounded by Γ_1 and Γ_3 (Fig.1 illustrates the two-dimensional case). Solution in Ω_1 and Ω_2 is obtained idenpendently by solving the problem of an arbitrary discontinuity on the curvilinear surface γ_{20} (the intersection of Γ_2 and the plane $t = 0$), and the incident wave surface Γ_2 is determined for $t > 0$ /5/. In what follows the solution in Ω_1 and Ω_2, and surface Γ_2 are assumed known. We have to construct the solution of Eqs. (1.2) in Ω_3 which satisfies conditions (1.1) and (1.3) on Γ_1, Γ_3, and at the same time determine the reflected wave surface Γ_3.

2. Relationships at the shock wave. In the case of regular reflection which obtains in the initial stage the unknown surface Γ_3 must pass through the known two-dimensional surface γ_0, the locus of Γ_1 and Γ_2 intersection points (Fig.1). Let surface γ_0 be parametrically defined: $t = t_0(\beta, \gamma)$, $\mathbf{x} = \mathbf{x}_0(\beta, \gamma)$ with t_0 and $\mathbf{x}_0$ being analytic functions of parameters β, γ, and $\mathbf{x}_{0\beta} \neq 0$, $\mathbf{x}_{0\gamma} \neq 0$, $\mathbf{x}_{0\beta} \times \mathbf{x}_{0\gamma} \neq 0$. We introduce in region Ω_3 new coordinates

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$\tau, \alpha, \beta, \gamma$ such that $\tau = 0$ corresponds to the shock wave surface, and $\alpha = 0$ to the rigid wall. The respective substitution of variables conforms to $t = t(\tau, \alpha, \beta, \gamma) = \tau + \alpha + t_0(\beta, \gamma)$, $\mathbf{x} = \mathbf{x}(\tau, \alpha, \beta, \gamma)$. Function $\mathbf{y} = \mathbf{x}(0, \alpha, \beta, \gamma)$ is determined by the solution of the Cauchy problem

$$\mathbf{y}_\alpha = \mathbf{H}, \quad \mathbf{y}|_{\alpha=0} = \mathbf{x}_0(\beta, \gamma)$$

Function $\mathbf{H}$ is selected so that equations $t = \alpha + t_0(\beta, \gamma)$, $\mathbf{x} = \mathbf{x}(0, \alpha, \beta, \gamma)$ parametrically define the shock wave surface Γ_3.

Let the equation of Γ_3 be of the form $\varphi_1(\mathbf{x}, t) = 0$. Then $\varphi_1(\mathbf{x}(0, \alpha, \beta, \gamma), t(0, \alpha, \beta, \gamma)) = 0$ identically with respect to α, β, γ. Differentiation of this equality with respect to α, β, γ yields the formulas

$$(\varphi_1)_t + \mathbf{x}_\alpha \nabla\varphi_1 = 0, \quad t_\beta(\varphi_1)_t + \mathbf{x}_\beta \nabla\varphi_1 = 0, \quad t_\gamma(\varphi_1)_t + \mathbf{x}_\gamma \nabla\varphi_1 = 0$$

Since the shock wave front normal to n and the quantity D_n are linked with φ_1 by the relations $\mathbf{n} = \nabla\varphi_1 |\nabla\varphi_1|^{-1}$, $D_n = -(\varphi_1)_t |\nabla\varphi_1|^{-1}$, we have the following equalities:

$$\mathbf{x}_\beta \mathbf{n} = t_\beta D_n, \quad \mathbf{x}_\gamma \mathbf{n} = t_\gamma D_n, \quad |\mathbf{n}| = 1, \quad \mathbf{x}_\alpha \mathbf{n} = D_n$$

of which the first three formulas enable us to determine $\mathbf{n}$ in the form

$$\mathbf{n} = \{(|\mathbf{m}|^2 - D_n^2 |\mathbf{k}|^2)^{1/2}\mathbf{m} + D_n(\mathbf{k} \times \mathbf{m})\} |\mathbf{m}|^{-2}, \qquad \mathbf{m} = \mathbf{x}_\beta \times \mathbf{x}_\gamma, \quad \mathbf{k} = t_\beta \mathbf{x}_\gamma - t_\gamma \mathbf{x}_\beta \tag{2.1}$$

where $\mathbf{k}$ is a vector tangent to the shock wave front by virtue of (2.1).

Function $\mathbf{H}$ must, therefore, satisfy the relation $\mathbf{Hn} = D_n$, where the normal $\mathbf{n}$ is defined by formula (2.1). Specific selection will be made in Sect.3. Writing the preceding formulas in the form

$$(\mathbf{x}_\beta - t_\beta \mathbf{u})\mathbf{n} = t_\beta(D_n - u_n), \quad (\mathbf{x}_\gamma - t_\gamma \mathbf{u})\mathbf{n} = t_\gamma(D_n - u_n), \quad |\mathbf{n}| = 1$$

we obtain for n another formula

$$\mathbf{n} = \{(|\mathbf{q}|^2 - v_n^2 |\mathbf{k}|^2)^{1/2}\mathbf{q} + v_n(\mathbf{k} \times \mathbf{q})\} |\mathbf{q}|^{-2} \tag{2.2}$$
$$\mathbf{q} = (\mathbf{x}_\beta - t_\beta \mathbf{u}) \times (\mathbf{x}_\gamma - t_\gamma \mathbf{u})$$

which is valid when $|\mathbf{q}| \geqslant |v_n \mathbf{k}|$ and the inequality $|\mathbf{m}| \geqslant |D_n \mathbf{k}|$ is satisfied in (2.1). If function $\mathbf{x}(0, \alpha, \beta, \gamma)$ is known, $\mathbf{x}(\tau, \alpha, \beta, \gamma)$ is determined similarly by solving the Cauchy problem

$$\mathbf{x}_\tau = \mathbf{G}, \quad \mathbf{x}|_{\tau=0} = \mathbf{x}(0, \alpha, \beta, \gamma)$$

Function $\mathbf{G}$ is selected so that the contact characteristic /3/ that passes at $\tau = 0$ through the cross section $\alpha = \text{const}$ of surface Γ_3 is parametrically specified by formulas $t = \tau + \alpha + t_0(\beta, \gamma)$, $\mathbf{x} = \mathbf{x}(\tau, \alpha, \beta, \gamma)$ for fixed α. If the equation of that characteristic is of the form $\varphi_2(x, t) = 0$, then as previously we obtain the equalities

$$(\varphi_2)_t + \mathbf{u}\nabla\varphi_2 = 0, \quad (\varphi_2)_t + \mathbf{x}_\tau \nabla\varphi_2 = 0, \quad t_\beta(\varphi_2)_t + \mathbf{x}_\beta \nabla\varphi_2 = 0, \quad t_\gamma(\varphi_2)_t + \mathbf{x}_\gamma \nabla\varphi_2 = 0$$

and their corollaries

$$(\mathbf{x}_\beta - t_\beta \mathbf{u})\mathbf{n}_2 = 0, \quad (\mathbf{x}_\gamma - t_\gamma \mathbf{u})\mathbf{n}_2 = 0, \quad (\mathbf{x}_\tau - \mathbf{u})\mathbf{n}_2 = 0, \quad |\mathbf{n}_2| = 1$$

Hence $\mathbf{n}_2 = \mathbf{q} |\mathbf{q}|^{-1}$ and function $\mathbf{G}$ must satisfy the relation $(\mathbf{G} - \mathbf{u})\mathbf{q} = 0$. The specific selection of $\mathbf{G}$ is effected in Sect.3. Since $\mathbf{kq} = \mathbf{kn} = 0$, vector $\mathbf{k}$ at points γ_0 is directed along the tangent to the line of intersection of the incident wave front and the wall, i.e. to the trace which the incident front leaves on the wall at instant $t = \text{const}$. A three-dimensional picture of this at instant $t = \text{const}$ appears in Fig.2 (γ_{2t}, γ_{3t}, γ_{0t} are cross sections of Γ_2, Γ_3, γ_0 at $t = \text{const}$).

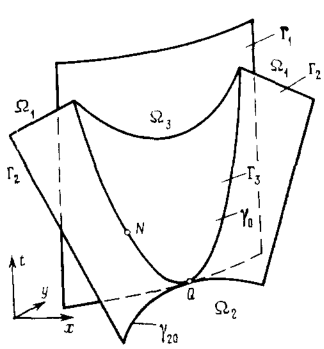

Fig.1

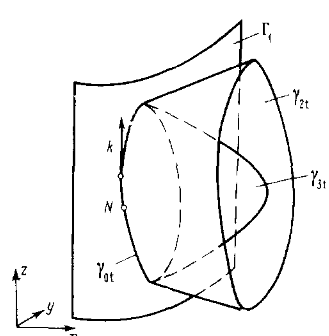

Fig.2

For analyzing the relationships at the shock wave it is convenient to use the vector $\mathbf{w}$

$$\mathbf{w} = (\mathbf{q} \times \mathbf{k}) |\mathbf{k}|^{-2} = \mathbf{u} - (\mathbf{uk}) |\mathbf{k}|^{-2}\mathbf{k} - (\mathbf{k} \times \mathbf{m}) |\mathbf{k}|^{-2}$$

which at points γ_0 represents the difference of projections of $\mathbf{u}$ on the plane normal to $\mathbf{k}$ and the trace motion velocity on the wall in a direction perpendicular to the trace. Since by virtue of (2.2) $w_n = \mathbf{wn} = -v_n$, hence on Γ_3

$$[\rho w_n] = 0,\ [p + \rho w_n^2] = 0,\ [\varepsilon + p\rho^{-1} + {}^1/_2\, w_n^2] = 0,\ [w_\sigma] = 0$$

From the corollaries of relations at discontinuity

$$|\mathbf{w}|^2 = |\mathbf{w}_1|^2 - (p - p_1)(v_1 + v),\ (\mathbf{w} - \mathbf{w}_1)^2 = (p - p_1) \times (v_1 - v)$$

where $v = \rho^{-1}$, and the quantities ahead of the front are denoted by subscript 1 and those behind it have no subscripts. For the angle θ of turn of $\mathbf{w}$ in the Π plane orthogonal to $\mathbf{k}$ we have at transition through the discontinuity $(\cos\theta = (\mathbf{w}\mathbf{w}_1)|\mathbf{w}|^{-1} \times |\mathbf{w}_1|^{-1})$ the relation

$$\sin^2\theta = (p - p_1)\{v_1 - v - v_1^2 |\mathbf{w}_1|^{-2}(p - p_1)\} \times \{|\mathbf{w}_1|^2 - (p - p_1)(v_1 + v)\}^{-1} \tag{2.3}$$

$$v = g(p, p_1, v_1)$$

where θ is the p-polar.

Function g defines the Hugoniot curve. In the θ, p plane curve (2.3) is determinate when $(v_1 - v)|\mathbf{w}_1|^2 v_1^{-2} \geqslant p - p_1 \geqslant 0$. Along that curve $|\sin\theta| < 1$ and θ vanishes when equality is reached in one of inequalities. Consequently θ reaches its maximum value θ_* (limit angle of turn) at some point $p = p_*$.

In what follows, we assume that the equations of state of the gas are such that the right-hand side of (2.3) is monotonic when $p < p_*$ and $p > p_*$ and that there exists a p_0 $(p_1 < p_0 < p_*)$ such that along the curve $|\mathbf{w}|^2 = |\mathbf{w}_1|^2 - (p - p_1)(v_1 + v) < c^2$ when $p > p_0$ and $|\mathbf{w}|^2 > c^2$ when $p < p_0$. A polytropic gas satisfies these conditions.

Using the identity

$$\sin^2\theta = |\mathbf{q} \times \mathbf{q}_1|^2 |\mathbf{q}|^{-2} |\mathbf{q}_1|^{-2}, \quad \mathbf{q} = \{(\mathbf{q}\mathbf{m})\,\mathbf{m} + (\mathbf{u}\mathbf{m})(\mathbf{k} \times \mathbf{m})\}\,|\mathbf{m}|^{-2}$$

we obtain from (2.3) the relation

$$\frac{h - h_1}{\{(1 + h^2|\mathbf{k}|^2)(1 + h_1^2|\mathbf{k}|^2)\}^{1/2}} = \left\{\frac{(p - p_1)(v_1 - v) - |\mathbf{k}|^2 v_1^2 |\mathbf{q}_1|^{-2}(p - p_1)^2}{|\mathbf{q}_1|^2 - (p - p_1)(v_1 + v)|\mathbf{k}|^2}\right\}^{1/2} \tag{2.4}$$

$$h = (\mathbf{u}\mathbf{m})(\mathbf{q}\mathbf{m})^{-1}, \quad h_1 = (\mathbf{u}_1\mathbf{m})(\mathbf{q}_1\mathbf{m})^{-1}$$

At point Q, where the shock wave front touches the wall, $(t_\beta = t_\gamma = 0,\ \mathbf{k} = 0)$, and (2.4) becomes the equation of the (p, u_n)-diagram of shock waves, which is determinate and monotonic for all values of p $(p_* \to \infty$ as $|\mathbf{k}| \to 0)$. In the region of determination (2.4)

$$|\mathbf{q}|^2 - v_n^2|\mathbf{k}|^2 = |\mathbf{k}|^2(|w|^2 - v_n^2) = |\mathbf{k}|^2(|\mathbf{w}_1|^2 - (p - p_1)(v_1 - v)^{-1} v_1^2) \geqslant 0$$

From (1.3) with allowance for the equalities

$$[\mathbf{x}_\beta \mathbf{u}] = (\mathbf{x}_\beta \mathbf{n})[u_n] = t_\beta D_n [u_n], \quad [\mathbf{x}_\gamma \mathbf{u}] = t_\gamma D_n [u_n]$$

we obtain two corollaries

$$[a] = [b] = 0 \quad (a = (\mathbf{u}\mathbf{x}_\beta) - t_\beta(\varepsilon + pv + {}^1/_2|\mathbf{u}|^2) \tag{2.5}$$
$$b = (\mathbf{u}\mathbf{x}_\gamma) - t_\gamma(\varepsilon + pv + {}^1/_2|\mathbf{u}|^2))$$

When $\mathbf{x}_\beta, \mathbf{x}_\gamma, t_\beta, t_\gamma$ at some point of surface Γ_3, the parameters of gas ahead of the front, and the quantity h $(h < h_*)$ behind it are known, it is possible to determine all parameters of gas behind the front, vector $\mathbf{n}$ and D_n.

In conformity with (2.1), at point Q where $t_\beta = t_\gamma = 0$, $\mathbf{n} = \mathbf{m}|\mathbf{m}|^{-1}$ we have $h = u_n|\mathbf{m}|^{-1}$. Equation (2.4) uniquely determines p in terms of h, and subsequently all remaining parameters of gas /3,4/. At points Γ_3, where $t_\beta^2 + t_\gamma^2 \neq 0$, Eq. (2.4) has for a given h two solutions p_-, p_+ $(p_- < p_* < p_+)$ (an analog of "weak" and "strong" shocks in the theory of steady motion). But, when $t_\beta^2 + t_\gamma^2 \to 0$, only p_- approaches p which is uniquely defined at point Q $(p_+ \to \infty$ as $|\mathbf{k}| \to 0)$. Hence on considerations of continuity of solution behind the front we uniquely select the "weak" shock. Using the Hugoniot adiabatic curve and the known value of p, we determine v, and using known p, v, p_1, v_1 we obtain v_n. The following identities apply:

$$(\mathbf{u}\mathbf{m}) = h|\mathbf{q}||\mathbf{m}|(1 + h^2|\mathbf{k}|^2)^{-1/2}, \quad |\mathbf{q}|^2 = |\mathbf{m}|^2 + 2\mathbf{k}(b\mathbf{x}_\beta - a\mathbf{x}_\gamma) - 2|\mathbf{k}|^2(\varepsilon + pv) - (t_\beta b - t_\gamma a)^2$$

$$\mathbf{u} = \{(\mathbf{u}\mathbf{m})\,\mathbf{m} + (a\mathbf{x}_\gamma - b\mathbf{x}_\beta) \times \mathbf{m} + (\varepsilon + pv + {}^1/_2|\mathbf{u}|^2)(\mathbf{k} \times \mathbf{m})\}\,|\mathbf{m}|^{-2}$$

by virtue of which it is possible to determine $\mathbf{u}$ in terms of the known quantities $h, a, b, p, v, \mathbf{x}_\beta, \mathbf{x}_\gamma, t_\beta, t_\gamma$. Then, $\mathbf{n}$ is determined using formula (2.2), and $D_n = v_n + (\mathbf{u}\mathbf{n})$. At points γ_0 the normal to the rigid wall coincides with vector $\mathbf{m}|\mathbf{m}|^{-1}$, hence $h = 0$ on γ_0. It is, consequently, possible to determine on γ_0 all parameters of gas behind the front, the normal $\mathbf{n}$, and D_n as analytic functions of variables β, γ at those points γ_0, where the regular reflection $(p < p_*)$ is possible.

3. Transformation of equations and boundary conditions. We introduce in region Ω_3 new coordinates, and the functions that define the passage to them are determined in conformity with Sect.2 as the solution of the Cauchy problem

$$\mathbf{y}_\alpha = D_n(|\mathbf{y}_\beta \times \mathbf{y}_\gamma|^2 - D_n^2|T_\beta \mathbf{y}_\gamma - T_\gamma \mathbf{y}_\beta|^2)^{-1/2}(\mathbf{y}_\beta \times \mathbf{y}_\gamma), \quad T_\alpha = 1 \tag{3.1}$$
$$\mathbf{y}|_{\alpha=0} = \mathbf{x}_0(\beta, \gamma), \quad T|_{\alpha=0} = t_0(\beta, \gamma)$$
$$\mathbf{x}_\tau = h(\mathbf{x}_\beta \times \mathbf{x}_\gamma), \quad t_\tau = 1, \quad \mathbf{x}|_{\tau=0} = \mathbf{y}|_{\tau=0}, \quad t|_{\tau=0} = T$$

where D_n is a function of $h, \mathbf{y}_\beta, \mathbf{y}_\gamma, T_\beta, T_\gamma$ and of quantities ahead of the front considered to be functions of $\mathbf{y}, T$, as determined in Sect.2. For the transformation of equations of motion we use the formulas

$$\nabla = \lambda\frac{\partial}{\partial\tau} + \mu\frac{\partial}{\partial\alpha} + I^{-1}\left\{(\mathbf{x}_\gamma - t_\gamma\mathbf{x}_\alpha) \times (\mathbf{x}_\tau - \mathbf{x}_\alpha)\frac{\partial}{\partial\beta} + (\mathbf{x}_\tau - \mathbf{x}_\alpha) \times (\mathbf{x}_\beta - t_\beta\mathbf{x}_\tau)\frac{\partial}{\partial\gamma}\right\}$$
$$\frac{d}{dt} = J\frac{\partial}{\partial\tau} + |\mathbf{m}|^{-2}\left\{\mathbf{u}(\mathbf{x}_\gamma \times \mathbf{m})\frac{\partial}{\partial\beta} - \mathbf{u}(\mathbf{x}_\beta \times \mathbf{m})\frac{\partial}{\partial\gamma}\right\}$$
$$\lambda = I^{-1}(\mathbf{x}_\beta - t_\beta\mathbf{x}_\alpha) \times (\mathbf{x}_\gamma - t_\gamma\mathbf{x}_\alpha),$$
$$\mu = -I^{-1}(\mathbf{x}_\beta - t_\beta\mathbf{x}_\tau) \times (\mathbf{x}_\gamma - t_\gamma\mathbf{x}_\tau),$$
$$I = (\mathbf{x}_\tau - \mathbf{x}_\alpha)\{(\mathbf{x}_\beta - t_\beta\mathbf{x}_\tau) \times (\mathbf{x}_\gamma - t_\gamma\mathbf{x}_\tau)\}, \quad J = (\mathbf{qm})|\mathbf{m}|^{-2}$$

The feasibility of passing to new variables depends on the quantity I not becoming zero or infinity. When $\tau = 0$ we have

$$I = -|\mathbf{m}|^{-2}v_n J^{-1}(|\mathbf{m}|^2 - D_n^2|\mathbf{k}|^2)^{-1/2}$$

This formula defines the required property of I, at least on γ_0.

The transformed equations (1.2) are schematically represented in the form

$$s_\tau = d_1 s_\beta + d_2 s_\gamma, \quad a_\tau = e_1\mathbf{V}_\beta + e_2\mathbf{V}_\gamma, \quad b_\tau = e_3\mathbf{V}_\beta + e_4\mathbf{V}_\gamma \tag{3.2}$$
$$\mathbf{U}_\tau = B\mathbf{U}_\alpha + D\mathbf{U}_\alpha + E_1\mathbf{V}_\beta + E_2\mathbf{V}_\gamma + H_1(\mathbf{x}_\alpha)_\beta + H_2(\mathbf{x}_\alpha)_\gamma$$
$$\mathbf{U} = \begin{Vmatrix} h \\ p \end{Vmatrix},$$
$$B = \frac{1}{J^2c^{-2} - |\lambda|^2}\begin{Vmatrix} (\lambda\mu), & I(\rho J^2|\mathbf{m}|^2)^{-1}(J^2|\mu|^2c^{-2} - |\lambda \times \mu|^2) \\ \rho J^2|\mathbf{m}|^2 I^{-1}, & (\lambda\mu) \end{Vmatrix}$$

where $\mathbf{V}$ is the vector solution whose components $s, a, b, h, p, x_\beta, x_\gamma, y_\beta, y_\gamma, z_\beta, z_\gamma$ and coefficients d_i, e_i, B, D, E_i, H_i are scalar, vector-valued, and matrix-valued analytic functions of variables $\mathbf{V}, \mathbf{x}_\alpha, \beta, \gamma, \alpha$. Matrix D vanishes when $\tau = 0$.

In conformity with Sect.2 boundary conditions are of the form

$$\tau = 0: \quad s = f_1(h, \mathbf{x}_\beta, \mathbf{x}_\gamma, \mathbf{x}, \alpha, \beta, \gamma), \quad a = f_2(\mathbf{x}_\beta, \mathbf{x}, \alpha, \beta, \gamma) \tag{3.3}$$
$$b = f_3(\mathbf{x}_\gamma, \mathbf{x}, \alpha, \beta, \gamma), \quad p = f_4(h, \mathbf{x}_\beta, \mathbf{x}_\gamma, \mathbf{x}, \alpha, \beta, \gamma); \quad \alpha = 0: \quad h = 0$$

where f_i are analytic functions of their arguments close to the values of the latter on γ_0. The dependence of f_i on $\mathbf{x}, \alpha, \beta, \gamma$ is linked to the appearance in the relations at the shock wave front of parameters of gas ahead of the front, which are known as functions of $\mathbf{x}, t$. The input problem is, thus, reduced to solving problem (3.1)—(3.3) followed by the inversion of representations that define transition to new variables.

4. Construction of solution in the class of formal power series. The problem formulated here is a generalization of the Goursat problem. The problem of derivation of its solution in the class of formal power series in variables $\tau, \alpha, \beta - \beta_1, \gamma - \gamma_1$, where $(0, 0, \beta_1, \gamma_1)$ are the coordinates of an arbitrary point N of surface γ_0, reduces to the calculation of derivatives of the solution at that point.

We write the last equation of system (3.2) in the form

$$\mathbf{U}_\tau = B_N\mathbf{U}_\alpha + \mathbf{F} \quad (B_N = B|_{\tau=\alpha=0,\ \beta=\beta_1,\ \gamma=\gamma_1}) \tag{4.1}$$

whose two correlations

$$B_N{}^j\mathbf{U}_{j,n-j} - \mathbf{U}_{0,n} + \sum_{k=1}^{j} B_N^{k-1}\mathbf{F}_{k-1,n-k} = \sum_{k=1}^{j} B_N^{k-1}(B_N\mathbf{U}_{1,0} - \mathbf{U}_{0,1} + \mathbf{F})_{k-1,n-k} = 0 \tag{4.2}$$
$$B_N^{j-n}\mathbf{U}_{j,n-j} - \mathbf{U}_{n,0} - \sum_{k=j+1}^{n} B_N^{k-1-n}\mathbf{F}_{k-1,n-k} = \sum_{k=j+1}^{n} B_N^{k-1-n}(\mathbf{U}_{0,1} - B_N\mathbf{U}_{1,0} - \mathbf{F})_{k-1,n-k} = 0$$
$$(f_{i,j} = \partial^{(i+j)}f/\partial\alpha^i\partial\tau^j)$$

are useful for calculating the derivatives.

If $\Lambda_1 \mathbf{U}_{0,n}$ is specified for $\alpha = 0$, and for $\tau = 0$ is $\Lambda_2 \mathbf{U}_{n,0}$, then by multiplying (4.2) by Λ_1, Λ_2, we respectively obtain a system of two equations in $\mathbf{U}_{j,n-1}$. In the considered here case $\Lambda_1 = (1, 0)$, $\Lambda_2 = (-\varkappa, 1)$ $(\varkappa = \partial f_4/\partial h\,|_{\tau=0})$. To clarify the question of the obtained system solvability, it is expedient to reduce matrix B_N to the diagonal form. The eigenvalues of matrix B_N are of the form

$$\nu_{1,2} = \{(\lambda\mu) \pm (J^2|\mu|^2 c^{-2} - |\lambda \times \mu|^2)^{1/2}\}\,(J^2c^{-2} - |\lambda|^2)^{-1}|_N$$

and real when $|\mathbf{w}_N| \geqslant c_N$ (subscript N denotes quantities at point N). Below, the case of strict inequality is called supersonic, and that of strict reverse inequality, subsonic.

In the neighborhood of point Q on γ_0 (i.e. for small t) we always have the supersonic case, while for large t transition to the subsonic case is possible (but not obligatory) depending on the geometry of incipient wave, its velocity, etc. Since all quantities on γ_0 are in conformity with Sect.2, determined prior to the solution of the problem, the "subsonic" and "supersonic" points, as well as the points at which the limit angle of turn of vector $\mathbf{w}$, are a priori known.

In the subsonic case $\nu_1 = \bar{\nu}_2$. For $\nu_1 \neq \nu_2$ matrix B_N reduces to the diagonal form

$$B_N = M^{-1}\chi M, \quad \chi = \mathrm{diag}\,\{\nu_2, \nu_1\}$$

It is convenient to introduce new unknown functions

$$\mathbf{r} = \left\|\begin{matrix} r \\ l \end{matrix}\right\| = M\mathbf{U} = \left\|\begin{matrix} h + Kp \\ -h + Kp \end{matrix}\right\|$$

$$K = -I\,(J^2|\mu|^2c^{-2} - |\lambda \times \mu|^2)^{1/2}\rho^{-1}(J\,|\mathbf{m}|)_N^{-2}$$

Equations (4.1) assume the form

$$r_\tau = \nu_2 r_\alpha - \psi^1, \quad l_\tau = \nu_1 l_\alpha + \psi^2, \quad \left\|\begin{matrix} -\psi^1 \\ \psi^2 \end{matrix}\right\| = M\mathbf{F} \tag{4.3}$$

Solving the equations obtained by multiplying (4.2) by Λ_1, Λ_2, we obtain the formulas

$$r_{j,n-j} = \Delta_n^{-1}\{\sum_{k=1}^{n} \nu_2^{n-j}\nu_1^{k-1-n}\psi^2_{k-1,n-k} + \sum_{k=1}^{j} \nu_2^{n-j+k-1}\nu_1^{-n}\psi^1_{k-1,n-k} + \tag{4.4}$$

$$\sum_{k=j+1}^{n} d\nu_2^{k-j-1}\psi^1_{k-1,n-k} + \nu_2^{n-j}\nu_1^{-n}(r_{0,n} - l_{0,n}) + \nu_2^{n-j}(l_{n,0} - dr_{n,0})\}$$

$$l_{j,n-j} = \Delta_n^{-1}\{\sum_{k=1}^{n} d\nu_2^{k-1}\nu_1^{-j}\psi^1_{k-1,n-k} + \sum_{k=1}^{j} d\nu_1^{k-j-1}\psi^2_{k-1,n-k} +$$

$$\sum_{k=j+1}^{n} \nu_2^{n}\nu_1^{k-n-j-1}\psi^2_{k-1,n-k} + d\nu_1^{-j}(r_{0,n} - l_{0,n}) + \nu_2\nu_1^{-j}(l_{n,0} - dr_{n,0})\}$$

$$\Delta_n = \nu_2^n\nu_1^{-n} - d, \quad d = (\varkappa K - 1)(\varkappa K + 1)^{-1}$$

These formulas are used for calculating the derivatives of order n with respect to variables τ, α of functions r, l in terms of derivatives of order $n-1$ of ψ^i and derivatives $(r_{0,n} - l_{0,n})_{\alpha=0}$, $(l_{n,0} - dr_{n,0})_{\tau=0}$ are known by virtue of (3.3). The necessary condition of solvability of equations for $r_{j,n-j}$, $l_{j,n-j}$ are of the form $\Delta_n \neq 0$, $n = 1, 2, \ldots$. If Δ_n vanishes, we obtain a contradiction to the assumption of existence an analytic solution of the problem with arbitrary data.

In the supersonic case $\nu_2 > \nu_1 > 0$, $|d| < 1$ and the conditions of solvability are always satisfied. In the subsonic case $\nu_2 = \bar{\nu}_1$, and K is an imaginary number. If $\nu_2\nu_1^{-1} = e^{2\pi i\sigma}$, $d = e^{2\pi i\delta}$ $(0 \leqslant \sigma \leqslant 1,\ 0 \leqslant \delta \leqslant 1$, the solvability condition is of the form $e^{2\pi in\sigma} - e^{2\pi i\delta} \neq 0$ for all positive integers n. The quantities Δ_n vanish on straight lines $\sigma = \delta n^{-1} + jn^{-1}$ $(j = 0, 1, \ldots, n-1)$ that lie in the unit square of the plane (σ, δ). Points of these straight lines for various n, j form everywhere in that square a compact set. Then in any neighborhood of point (σ, δ) with $\Delta_n \neq 0$ $(n = 1, 2, \ldots)$ there are points at which certain $\Delta_k = 0$. This property can be interpreted as instability of the subsonic problem in the class of analytic functions. Since on γ_0, σ and δ are continuous functions of β and γ, it is possible to maintain that when $\sigma \neq \mathrm{const}$, $\delta \neq \mathrm{const}$ there are points on γ_0 at which $\Delta_k = 0$. Consequently the problem of regular reflection has, generally, no analytic solution in the subsonic case.

Below, we consider the supersonic case. It is convenient to transform the boundary conditions that link r and l to the form

$$(r - l)_{\alpha=0} = 0, \quad (l - dr)_{\tau=0} = g_1(r, \mathbf{x}_\beta, \mathbf{x}_\gamma, \mathbf{x}, \alpha, \beta, \gamma) \tag{4.5}$$

where g_1 is an analytic function of its arguments, such that $(\partial g_1/\partial r)_N = 0$.

Lemma. A unique solution of problem (3.1)—(3.3) exists in the class of formal power series in variables $\tau, \alpha, \beta-\beta_1, \gamma-\gamma_1$.

Proof is by induction over the total number n of differentiation with respect to τ and α. In accordance with Sect.2 all sought functions are known on γ_0 when $\tau=\alpha=0$. Differentiating them with respect to β and γ, we obtain all derivatives with respect to these variables. If all derivatives with the over-all $n-1$ of differentiation with respect to τ, α, the derivatives of r and l of order n with respect to τ and α are determined using (4.4) and the formulas obtained by differentiating (4.4) with respect to β and γ. Derivatives of remaining functions are determined by equations obtained by differentiating (3.1)—(3.3).

5. Existence of analytic solution of the problem in the supersonic case.

Convergence of formal power series is proved by costructing majorants of solutions that are obtained by solving the auxiliary majorant problem. As a preliminary, the nonlinear equations (3.1) in $\mathbf{x}$ and $\mathbf{y}$ are reduced to quasi-linear equations in $\mathbf{x}$ and $\mathbf{y}$ by extending them to derivatives, and boundary conditions of the problem are reduced to a homogeneous form by substituting the unknown functions (subscript unity denotes new notation) $\mathbf{y}_1=\mathbf{y}-\mathbf{x}_0(\beta,\gamma), \mathbf{x}_1=\mathbf{x}-\mathbf{y}$ and the respective substitutions of derivatives of these functions; further

$$s_1=s-f_1, \quad a_1=a-f_2, \quad b_1=b-f_3, \quad r_1=r-g_1(1-d)^{-1}, \quad l_1=l-g_1(1-d)^{-1}$$

For the transformation of quantities we obtain the homogeneous boundary conditions

$$\begin{aligned}&\alpha=0: \quad \mathbf{y}_1=\mathbf{y}_{1\beta}=\mathbf{y}_{1\gamma}=0, \quad r_1-l_1=0\\&\tau=0: \quad \mathbf{x}_1=\mathbf{x}_{1\beta}=\mathbf{x}_{1\gamma}=\mathbf{x}_{1\alpha}=0, \quad a_1=b_1=s_1=l_1-dr_1=0\end{aligned} \tag{5.1}$$

Formulas of the form (4.4) also apply to transformed ψ^1 and ψ^2. These formulas and the transformed equations imply that the problem with boundary conditions of form (5.1), where the equality sign is replaced by the majorizing relation ($s_1|_{\tau=0}\gg 0$, etc.) can be taken as the majorizing problem. We recall that the relation $f\gg g$ means that the coefficients of expansion of function f in series in powers of its arguments are not lower than the absolute values of respective coefficients of g.

Equations of the majorizing problem are obtained as follows. Coefficients at derivatives in the right-hand sides of transformed equations and in the expressions of transformed functions ψ^1, ψ^2 are replaced by their majorants, with the retention of the property of some coefficients vanishing at point N. Functions that satisfy "like" equations are majorized by one majorant.

Let Y be the majorant for all components $y_1, y_{1\beta}, y_{1\gamma}$, X the majorant for components x_1, $x_{1\beta}, x_{1\gamma}$, Z the majorant for components $x_{1\alpha}$, S the majorant for function s_1, R and L the majorants for r_1 and l_1, and A the majorant for a_1 and b_1. The majorant system is of the form

$$\begin{aligned}&Y_\alpha=F_1\Psi_1, \quad X_\tau=F_2(Y_\tau-\Psi_1), \quad Z_\tau=F_3(L_\alpha+R_\tau+\Psi_1+\Psi_2)\\&S_\tau=F_4(L_\alpha+R_\tau+\Psi_1+\Psi_2), \quad A_\tau=F_5\Psi_1,\\&\nu_2R_\alpha=R_\tau+F_6(\Psi_1+\Psi_2)\\&L_\tau=\nu_1L_\alpha+F_6(\Psi_1+\Psi_2), \quad \xi=\eta_1\tau+\eta_2\alpha+\beta-\beta_1+\gamma-\gamma_1,\\&\Psi_1=\Sigma_\beta+\Sigma_\gamma+1, \quad \Psi_2=(\Sigma+\xi)(\Sigma_\tau+\Sigma_\alpha), \quad \Sigma=(Y+X+Z+S+A+R+L).\end{aligned} \tag{5.2}$$

where η_1 and η_2 are constants which will be defined later, and F_i are the majorants of coefficients of input equations.

The solution of system (5.2) must vanish at point N. Hence the majorants of coefficients in that point neighborhood can be taken in the form

$$F_i=K_i(1-(\Sigma+\xi)M_i)^{-1}$$

with appropriately selected constants K_i, M_i. Formulas similar to (4.4) can be obtained for Eqs.(5.2), from which follows that, if we set in boundary conditions (5.1) $d=1$, the obtained problem is also a majorant for the input problem.

We seek a particular solution of Eqs.(5.2) that depends only on ξ, vanishes at point $\xi=0$, and such that $R\equiv L$. By virtue of (5.2) the last requirement provides the following relation between η_1 and η_2: $\eta_1={}^1/_2(\nu_2+\nu_1)\eta_2$. The system of equations for finding the particular solution $X(\xi), Z(\xi), S(\xi), A(\xi), Y(\xi), R(\xi)\equiv L(\xi)$ is of the form

$$\begin{aligned}&\eta_2Y'=F_1(2\Sigma'+1), \quad \varkappa_1\eta_2X'=F_2(\varkappa_1n_2Y'+2\Sigma'+1)\\&\varkappa_1\eta_2Z'=F_3(\eta_2L'+\varkappa_1\eta_2R'+\Psi_3), \quad \varkappa_1\eta_2A'=F_5(2\Sigma'+1)\\&\varkappa_1\eta_2S'=F_4(\eta_2L'+\varkappa_1\eta_2R'+\Psi_3), \quad \varkappa_2\eta_2R'=F_6\Psi_3\\&\varkappa_1={}^1/_2(\nu_2+\nu_1), \quad \varkappa_2={}^1/_2(\nu_2-\nu_1), \quad \Psi_3=((\Sigma+\xi)\eta_2(1+\varkappa_1)+2)\Sigma'+1\end{aligned}$$

The substitution of expressions for Y', R', L' from the first and last of equations into the right-hand sides of the remaining ones together with the linear combination of equations yield for function Σ the equation

$$\eta_2\Sigma' = (F_1 + \varkappa_1^{-1}F_2(\varkappa_1F_1 + 1) + \varkappa_1^{-1}F_5)(2\Sigma' + 1) + (\varkappa_1^{-1}(F_3 + F_4)((1 + \varkappa_1)\varkappa_2^{-1}F_6 + 1) + 2\varkappa_2^{-1}F_6)(\Psi_3 - 1)$$

In the neighborhood of point $\Sigma = \xi = 0$ this equation can be solved for Σ'. For this a fairly large parameter η_2 is selected, namely

$$\eta_2 > 2(K_1 + \varkappa_1^{-1}K_2(\varkappa_1K_1 + 1) + \varkappa_1^{-1}K_5) + 2(\varkappa_1^{-1}(K_3 + K_4)((1 + \varkappa_1)\varkappa_2^{-1}K_6 + 1) + 2\varkappa_2^{-1}K_6)$$

The equation for the determination of Σ now assumes the form

$$\Sigma' = \Phi(\Sigma, \xi) \tag{5.3}$$

with function Φ of the majorant type (with positive coefficients of expansion in series in powers of its arguments in the neighborhood of point $\Sigma = \xi = 0$). By the Cauchy–Kovalevsky theorem there exists an analytic solution of Eq.(5.3) that is, also, a function of the majorant type, and $\Sigma(0) = 0$. Using the known function Σ we determine functions $Y, X, Z, S, A, R = L$ which are also of the majorant type. We, thus, obtain the particular solution of the majorant system (5.2), which satisfies the conditions

$$Y|_{\alpha=0} \gg 0, \quad X|_{\tau=0} \gg 0, \quad Z|_{\tau=0} \gg 0, \quad S|_{\tau=0} \gg 0, \quad A|_{\tau=0} \gg 0, \quad R = L$$

A successive calculations of coefficients of expansion of this solution in power series will show that these series majorize the power series constructed in Sect.4.

6. The one-sheeted mapping of $(\tau, \alpha, \beta, \gamma) \to (t, x, y, z)$. The Jacobian of that mapping is finite and nonzero at points of γ_0. This enables us to use locally in the neighborhood of every point of γ_0 the implicit function theorem. For small τ, α (i.e. in the γ_0 neighborhood) the mapping is one-sheeted, hence the equalities

$$t(\tau_1, \alpha_1, \beta_1, \gamma_1) = t(\tau_2, \alpha_2, \beta_2, \gamma_2), \quad \mathbf{x}(\tau_1, \alpha_1, \beta_1, \gamma_1) = \mathbf{x}(\tau_2, \alpha_2, \beta_2, \gamma_2)$$

imply the equalities

$$\tau_1 = \tau_2, \ \alpha_1 = \alpha_2, \quad \beta_1 = \beta_2, \quad \gamma_1 = \gamma_2$$

By the definition of function t we have $\alpha_1 + \tau_1 + t_0(\beta_1, \gamma_1) = \alpha_2 + \tau_2 + t_0(\beta_2, \gamma_2)$. In conformity with (3.1) we have for functions $\mathbf{x}(\tau, \alpha, \beta, \gamma)$ valid the following formula

$$\mathbf{x}(\tau, \alpha, \beta, \gamma) = \mathbf{x}_0(\beta, \gamma) + \int_0^\alpha \mathbf{H}(0, \xi, \beta, \gamma)\,d\xi + \int_0^\tau \mathbf{G}(\eta, \alpha, \beta, \gamma)\,d\eta \tag{6.1}$$

where $\mathbf{H}$ and $\mathbf{G}$ coincide with the right-hand sides of Eqs.(3.1), and it is possible to substitute in $\mathbf{H}$ function $\mathbf{x}$ for $\mathbf{y}$ owing to their equality when $\tau = 0$.

Consider the neighborhood of γ_0 in which $|\mathbf{H}| < C_1$, $|\mathbf{G}| < C_1$ (C_1 is a positive constant).From (6.1) we have the inequality

$$|\mathbf{x}(\tau, \alpha, \beta, \gamma) - \mathbf{x}_0(\beta, \gamma)| < C_1(\tau + \alpha)$$

Then the following two inequalities are also satisfied:

$$|\mathbf{x}_0(\beta_1, \gamma_1) - \mathbf{x}_0(\beta_2, \gamma_2)| < C_1(\tau_1 + \alpha_1 + \tau_2 + \alpha_2), \quad |t_0(\beta_1, \gamma_1) - t_0(\beta_2, \gamma_2)| < \tau_1 + \alpha_1 + \tau_2 + \alpha_2$$

The specified mapping $t = t_0(\beta, \gamma)$, $\mathbf{x} = \mathbf{x}_0(\beta, \gamma)$ is such that from the last inequalities we have

$$|\beta_1 - \beta_2| + |\gamma_1 - \gamma_2| < C_2(\tau_1 + \alpha_1 + \tau_2 + \alpha_2)$$

with some positive constant C_2. By virtue of (6.1) the relation $\mathbf{x}(\tau_1, \alpha_1, \beta_1, \gamma_1) = \mathbf{x}(\tau_2, \alpha_2, \beta_2, \gamma_2)$ can be written in the form

$$\begin{aligned}
&\mathbf{x}_0(\beta_1, \gamma_1) - \mathbf{x}_0(\beta_2, \gamma_2) - (\alpha_2 - \alpha_1)\mathbf{H}(0, 0, \beta_2, \gamma_2) - \\
&(\tau_2 - \tau_1)\mathbf{G}(0, \alpha_2, \beta_2, \gamma_2) = -\int_0^{\alpha_1} (\mathbf{H}(0, \xi, \beta_1, \gamma_1) - \mathbf{H}(0, \xi, \beta_2, \gamma_2))\,d\xi - \\
&\int_0^{\tau_1} (\mathbf{G}(\eta, \alpha_1, \beta_1, \gamma_1) - \mathbf{G}(\eta, \alpha_2, \beta_2, \gamma_2))\,d\eta + \int_{\alpha_1}^{\alpha_2} (\mathbf{H}(0, \xi, \beta_2, \gamma_2) - \\
&\mathbf{H}(0, 0, \beta_2, \gamma_2))\,d\xi + \int_{\tau_1}^{\tau_2} (\mathbf{G}(\eta, \alpha_2, \beta_2, \gamma_2) - \mathbf{G}(0, \alpha_2, \beta_2, \gamma_2))\,d\eta
\end{aligned}$$

from which we obtain the inequality

$$| (\mathbf{x}_{0\beta} (\beta_2, \gamma_2) - t_\beta (\beta_2, \gamma_2) \mathbf{G} (0, 0, \beta_2, \gamma_2)) (\beta_1 - \beta_2) + (\mathbf{x}_{0\gamma} (\beta_2, \gamma_2) - t_\gamma (\beta_2, \gamma_2) \mathbf{G} (0, 0, \beta_2, \gamma_2)) (\gamma_1 - \gamma_2) + (\mathbf{H} (0, 0, \beta_2, \gamma_2) - \mathbf{G} (0, 0, \beta_2, \gamma_2)) (\alpha_1 - \alpha_2) | \leqslant C_3 (\tau_1 + \alpha_1 + \tau_2 + \alpha_2) (| \beta_1 - \beta_2 | + | \gamma_1 - \gamma_2 | + | \alpha_1 - \alpha_2 |)$$

with some constant $C_3 > 0$. From the obtained inequality we obtain for small $\tau_1, \tau_2, \alpha_1, \alpha_2$ the required equalities by virtue of the linear independence of vectors $\mathbf{x}_\beta - t_\beta \mathbf{G}, \mathbf{x}_\gamma - t_\gamma \mathbf{G}, \mathbf{H} - \mathbf{G}$ at points of γ_0. The one-sheeted property of mapping is proved. In the indicated neighborhood of γ_0 the mapping can be inverted yielding a solution in the form of analytic functions of $\mathbf{x}, t$.

The region in which the problem was solved comprises the band $0 \leqslant t < t_0$ with some $t_0 > 0$. The obtained solution, thus, defines the initial stage of shock wave reflection from the wall. An analytic solution of the problem of regular reflection that is uniquely determined in some neighborhood of the incident wave trace moving along the wall was obtained for large t under the condition of analyticity of function that define the gas flow in the neighborhood of the incident wave front. The above reasoning shows that, when the projection on the Π plane of gas velocity relative to the trace behind the reflected shock wave front becomes subsonic (at small t it is higher than the local speed of sound), the conditions of solvability of the problem in the class of analytic functions are not satisfied, which implies the appearance of solution singularities. In the case of convergence of the derived series and one-to-one mapping $(\tau, \alpha, \beta, \lambda) \to (t, x, y, z)$ the obtained solution defines the total flow of gas behind the reflected wave, and not only in the trace neighborhood. The transition from a regular shock wave reflection to the irregular one requires investigation.

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